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STUDY OF GENERALIZED LEGENDRE-APPELL POLYNOMIALS VIA FRACTIONAL OPERATORS

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Abstract. In this article, the operational definitions and integral representations are combined to introduce new families of the generalized Legendre and generalized Legendre-Appell polynomials. The explicit summation formulae, determinant definitions and recurrence relations for the generalized Legendre-Appell polynomials are derived by making use of the integral transforms and appropriate operational rules. An analogous study of these results for the generalized Legendre-Bernoulli, Legendre-Euler and Legendre-Genocchi polynomials is presented. Several identities for these polynomials are also derived by employing appropriate operational definitions.

Keywords: Appell polynomials, Legendre polynomials, Legendre-Appell polynomials; fractional operators, operational rules, determinant definition.

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1. Introduction

Differentiation and integration are usually regarded as discrete operations, in the sense that a function can be differentiated or integrated once, twice, or any number of times. However, in some circumstances it is useful to evaluate a fractional derivative. Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation operator. The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives, see for example [8].

One of the important classes of polynomial sequences is the class of Appell polynomial sequences [2], which arises in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. The generating function for the Appell polynomial sequences is given by

$$A(y,t) := A(t)e^{yt} = \sum_{n=0}^{\infty} A_n(y)\frac{t^n}{n!}. \tag{1}$$

The power series $A(t)$ is then given by

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!},\ A_0 \neq 0, \tag{2}$$

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with A_i $(i = 0, 1, 2, \cdots)$ being real coefficients and the function $A(t)$ is an analytic function at $t = 0$. The class of Appell sequences contains a large number of classical polynomial sequences such as the Bernoulli, Euler, Genocchi, Hermite and Laguerre polynomials *etc.*

The generating function for the Bernoulli polynomials $B_n(y)$ is given by [9, p. 36]

$$\left(\frac{t}{e^t - 1}\right) e^{yt} = \sum_{n=0}^{\infty} B_n(y) \frac{t^n}{n!}, \qquad |t| < 2\pi, \tag{3}$$

where $B_k := B_k(0)$ is the k^{th} Bernoulli number.

The generating function for the Euler polynomials $E_n(y)$ is given by [9, p. 40]

$$\left(\frac{2}{e^t + 1}\right) e^{yt} = \sum_{n=0}^{\infty} E_n(y) \frac{t^n}{n!}, \qquad |t| < \pi, \tag{4}$$

where $E_k := 2^k E_k\left(\frac{1}{2}\right)$ is the k^{th} Euler number.

The generating function for the Genocchi polynomials $G_n(y)$ is given by [14]

$$\left(\frac{2t}{e^t + 1}\right) e^{yt} = \sum_{n=0}^{\infty} G_n(y) \frac{t^n}{n!}, \qquad |t| < \pi, \tag{5}$$

where $G_k := G_k(0)$ is the k^{th} Genocchi number.

The Appell polynomials and related members are being characterized from different aspects, for this see [10, 13, 15, 16, 17, 18, 19, 20].

Various generalizations of the special functions of mathematical physics have witnessed a significant evolution during the recent years. This further advancement in the theory of special functions serves as an analytic foundation for the majority of problems in mathematical physics that have been solved exactly and find broad practical applications. An important development in the theory of generalized special functions is the introduction of multi-index and multi-variable special functions.

To give an example, we consider the 2-variable Legendre polynomials $S_n(x, y)$, introduced by Dattoli and Ricci [7]. These polynomials are of intrinsic mathematical importance and also have applications in physics. The generating equation for the Legendre polynomials is given by [7]

$$e^{yt}\ J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}, \tag{6}$$

where $J_0(xt)$ is the 0^{th} order ordinary Bessel function of the first kind [1] defined by the following series expression:

$$J_n(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k\ (\sqrt{x})^{n+2k}}{k!\ (n+k)!}. \tag{7}$$

We note that

$$\exp(-\alpha D_x^{-1}) = J_0(2\sqrt{\alpha x}), \qquad D_x^{-n}\{1\} := \frac{x^n}{n!}, \tag{8}$$

is the inverse derivative operator.

The polynomials $S_n(x, y)$ are also defined by the following operational rule:

$$\exp\left(D_x^{-1} \frac{\partial^2}{\partial y^2}\right) \{y^n\} = S_n(x, y). \tag{9}$$

Next, we recall the 2-variable Legendre-Appell polynomials (LeAP) ${}_SA_n(x,y)$, were introduced by Khan and Raza [11]. The generating equation for the Legendre-Appell polynomials ${}_SA_n(x,y)$ is given by

$$A(t)\exp(yt)J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} {}_SA_n(x,y)\frac{t^n}{n!}, \tag{10}$$

or, equivalently

$$A(t)\exp(yt)\exp(D_x^{-1}t^2) = \sum_{n=0}^{\infty} {}_SA_n(x,y)\frac{t^n}{n!}. \tag{11}$$

From equation (11), we have

$$\frac{\partial^2}{\partial y^2}{}_SA_n(x,y) = n(n-1){}_SA_{n-2}(x,y) \quad \text{and} \quad \frac{\partial}{\partial D_x^{-1}}{}_SA_n(x,y) = n(n-1){}_SA_{n-2}(x,y), \tag{12}$$

which consequently gives

$$\frac{\partial^2}{\partial y^2}{}_SA_n(x,y) = \frac{\partial}{\partial D_x^{-1}}{}_SA_n(x,y). \tag{13}$$

Also, from generating functions (11) and (1), it follows that

$${}_SA_n(0,y) = A_n(y), \tag{14}$$

solving equation (13) with condition (14) yields the following operational rule for the LeAP ${}_SA_n(x,y)$:

$${}_SA_n(x,y) = \exp\left(D_x^{-1}\frac{\partial^2}{\partial y^2}\right)\{A_n(y)\}. \tag{15}$$

The Euler's integral forms the basis of new generalizations of special polynomials. Dattoli *et al.* in [8] used the Euler integral to find the operational definitions and the generating relations for the generalized and new forms of special polynomials.

The Euler integral is given by [17, p. 218]

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-at}t^{\nu-1}dt,\ \min\{\mathrm{Re}(\nu),\mathrm{Re}(a)\} > 0, \tag{16}$$

which consequently yields the following [8]:

$$\left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\ e^{t\frac{\partial}{\partial x}}f(x)dt = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\ f(x+t)dt. \tag{17}$$

For the second order derivatives, we have the following formula:

$$\left(\alpha - \frac{\partial^2}{\partial x^2}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\ e^{t\frac{\partial^2}{\partial x^2}}f(x)dt. \tag{18}$$

In this paper, the generalized forms of the Legendre polynomials and Legendre-Appell polynomials are introduced and studied via fractional operators. The explicit summation formulae, determinant definitions and recurrence relations are derived for the generalized Legendre-Appell polynomials. The corresponding results for the generalized Legendre-Bernoulli, Legendre-Euler and Legendre-Genocchi polynomials are also deduced.

2. Generalized Legendre-Appell polynomials

To give the operational rule and generating equation for the generalized Legendre polynomials, we prove the following results:

Theorem 2.1. *For the generalized Legendre polynomials ${}_{\nu}S_n(x,y;\alpha)$, the following operational rule holds true:*

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} y^n = {}_{\nu}S_n(x,y;\alpha). \tag{19}$$

Proof. Replacing a by $\left(\alpha - \left(D_x^{-1}\frac{\partial^2}{\partial y^2}\right)\right)$ in integral (16) and then operating the resultant equation on y^n, we find

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} \{y^n\} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(tD_x^{-1}\frac{\partial^2}{\partial y^2}\right) y^n dt, \tag{20}$$

which in view of equation (9) becomes

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} \{y^n\} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}S_n(xt,y)dt. \tag{21}$$

The transform on the r.h.s of equation (21) defines a new family of polynomials. Denoting this special family of polynomials by ${}_{\nu}S_n(x,y;\alpha)$ and naming it as the generalized Legendre polynomials, so that we have

$${}_{\nu}S_n(x,y;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}S_n(xt,y)dt. \tag{22}$$

In view of equations (21) and (22), assertion (19) follows. □

Theorem 2.2. *For the generalized Legendre polynomials ${}_{\nu}S_n(x,y;\alpha)$, the following generating function holds true:*

$$\frac{\exp(yu)}{(\alpha - (D_x^{-1}u^2))^{\nu}} = \sum_{n=0}^{\infty} {}_{\nu}S_n(x,y;\alpha)\frac{u^n}{n!}. \tag{23}$$

Proof. Multiplying both sides of equation (22) by $\frac{u^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^{\infty} {}_{\nu}S_n(x,y;\alpha)\frac{u^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}S_n(xt,y)\frac{u^n}{n!}\, dt. \tag{24}$$

Using equation (6) in the r.h.s. of equation (24), it follows that

$$\sum_{n=0}^{\infty} {}_{\nu}S_n(x,y;\alpha)\frac{u^n}{n!} = \frac{\exp(yu)}{\Gamma(\nu)}\int_0^\infty e^{-\left(\alpha - D_x^{-1}u^2\right)t}t^{\nu-1}dt. \tag{25}$$

Making use of equation (16) in the r.h.s. of the above equation, assertion (23) follows. □

Next, we derive the operational definition and generating equation for the generalized Legendre-Appell polynomials by proving the following results:

Theorem 2.3. *For the generalized Legendre-Appell Polynomials ${}_{\nu S}A_n(x,y;\alpha)$, the following operational definition holds true:*

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu}\{A_n(y)\} = {}_{\nu S}A_n(x,y;\alpha). \tag{26}$$

Proof. Replacing a by $\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)$ in integral (16) and operating it on $A_n(y)$, we find

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu}\{A_n(y)\} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(tD_x^{-1}\frac{\partial^2}{\partial y^2}\right)\ A_n(y)dt, \tag{27}$$

which in view of equation (15) becomes

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} A_n(y) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_S A_n(xt,y)dt. \tag{28}$$

The transform on the r.h.s of equation (28) defines a new family of polynomials. Denoting this special family of polynomials by ${}_{\nu S}A_n(x,y;\alpha)$ and naming it as the generalized Legend-Appell polynomials, so that we have

$${}_{\nu S}A_n(x,y;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_S A_n(xt,y)dt. \tag{29}$$

In view of equations (28) and (29), assertion (26) follows. □

Theorem 2.4. *For the generalized Legendre-Appell polynomials ${}_{\nu S}A_n(x,y;\alpha)$, the following generating function holds true:*

$$\frac{A(u)\ \exp(yu)}{(\alpha - (D_x^{-1}u^2))^\nu} = \sum_{n=0}^\infty {}_{\nu S}A_n(x,y;\alpha)\frac{u^n}{n!}. \tag{30}$$

Proof. Multiplying both sides of equation (29) by $\frac{u^n}{n!}$ and summing over n, we find

$$\sum_{n=0}^\infty {}_{\nu S}A_n(x,y;\alpha)\frac{u^n}{n!} = \sum_{n=0}^\infty \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_S A_n(xt,y)\frac{u^n}{n!}\ dt. \tag{31}$$

Using generating function (11) in the r.h.s. of equation (31), it follows that

$$\sum_{n=0}^\infty {}_{\nu S}A_n(x,y;\alpha)\frac{u^n}{n!} = \frac{A(u)\ \exp(yu)}{\Gamma(\nu)}\int_0^\infty e^{-\left(\alpha-(D_x^{-1}u^2)\right)t}t^{\nu-1}dt, \tag{32}$$

which on use of integral (16) in the r.h.s. yields assertion (30).

□

Remark 2.1. We remark that, for $\alpha = \nu = 1$ and $x \to D_x^{-1}$, the generalized Legendre polynomials ${}_{\nu}S_n(x,y;\alpha)$ and generalized Legendre-Appell polynomials ${}_{\nu S}A_n(x,y;\alpha)$ reduce to Legendre polynomials $S_n(x,y)$ and Legendre-Appell polynomials ${}_S A_n(x,y)$, respectively.

Next, we derive an explicit summation formula for the generalized Laguerre-Appell polynomials ${}_{\nu S}A_n(x,y;\alpha)$ by proving the following result:

Theorem 2.5. *For the generalized Legendre-Appell polynomials ${}_\nu SA_n(x, y, ; \alpha)$, the following explicit summation formula in terms of the generalized Legendre polynomials ${}_\nu S_n(x, y; \alpha)$ and Appell polynomials $A_n(y)$ holds true:*

$$ {}_\nu SA_n(x, y; \alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ A_r(w)\ {}_\nu S_{n-k-r}(x, y; \alpha). \tag{33} $$

Proof. Consider the product of generating functions (1) and (23) in the following form:

$$ A(t)e^{wt}\left(\alpha - (D_x^{-1}t^2)\right)^{-\nu} \exp(yt) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A_r(w)\ {}_\nu S_n(x, y; \alpha)\frac{t^{n+r}}{n!\ r!}. \tag{34} $$

Replacing n by $n - r$ in the r.h.s. of equation (34) and shifting the first exponential to the r.h.s., it follows that

$$ A(t)\left(\alpha - (D_x^{-1}t^2)\right)^{-\nu} \exp(yt) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r}(-1)^k w^k A_r(w)\ {}_\nu S_{n-r}(x, y; \alpha)\frac{t^n}{n!}, \tag{35} $$

which on replacing n by $n - k$ gives

$$ A(t)\left(\alpha - (D_x^{-1}t^2)\right)^{-\nu} \exp(yt) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ A_r(w)\ {}_\nu S_{n-k-r}(x, y; \alpha)\frac{t^n}{n!}. \tag{36} $$

Finally, using generating function (30) in the l.h.s. of equation (36) and then equating the coefficients of like powers of t in the resultant equation, assertion (34) follows.

□

Remark 2.2. By taking $A(u) = \left(\frac{u}{e^u-1}\right)$ and $A_n(u) = B_n(u)$ in equations (26), (30) and (33), we find that for the generalized Legendre-Bernoulli polynomials ${}_\nu SB_n(x, y; \alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$ \left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} \{B_n(y)\} = {}_\nu SB_n(x, y; \alpha), \tag{37} $$

$$ \left(\frac{u}{e^u - 1}\right)\frac{\exp(yu)}{(\alpha - (D_x^{-1}u^2))^\nu} = \sum_{n=0}^{\infty} {}_\nu SB_n(x, y; \alpha)\frac{u^n}{n!}, \tag{38} $$

$$ {}_\nu SB_n(x, y; \alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ B_r(w)\ {}_\nu S_{n-k-r}(x, y; \alpha). \tag{39} $$

Remark 2.3. Taking $A(u) = \left(\frac{2}{e^u+1}\right)$ and $A_n(u) = E_n(u)$ in equations (26), (30) and (33), we find that for the generalized Legendre-Euler polynomials ${}_\nu SE_n(x, y; \alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$ \left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} \{E_n(y)\} = {}_\nu SE_n(x, y; \alpha), \tag{40} $$

$$ \left(\frac{2}{e^u + 1}\right)\frac{\exp(yu)}{(\alpha - (D_x^{-1}u^2))^\nu} = \sum_{n=0}^{\infty} {}_\nu SE_n(x, y; \alpha)\frac{u^n}{n!}, \tag{41} $$

$$ {}_\nu SE_n(x, y; \alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ E_r(w)\ {}_\nu S_{n-k-r}(x, y; \alpha). \tag{42} $$

Remark 2.4. Taking $A(u) = \left(\frac{2u}{e^u+1}\right)$ and $A_n(u) = G_n(u)$ in equations (26), (30) and (33), we find that for the generalized Legendre-Genocchi polynomials ${}_SG_{n,\nu}(x,y;\alpha)$, the following operational rule, generating equation and explicit summation formula hold true:

$$\left(\alpha - D_x^{-1}\left(\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu} G_n(y) = {}_{\nu S}G_n(x,y;\alpha), \tag{43}$$

$$\left(\frac{2u}{e^u+1}\right)\frac{\exp(yu)}{(\alpha-(D_x^{-1}u^2))^{\nu}} = \sum_{n=0}^{\infty} {}_{\nu S}G_n(x,y;\alpha)\frac{u^n}{n!}, \tag{44}$$

$${}_{\nu S}G_n(x,y;\alpha) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-1)^k\ w^k\ G_r(w)\ {}_{\nu}S_{n-k-r}(x,y;\alpha). \tag{45}$$

In the next section, we derive the determinant forms and recurrence relations for the generalized Legendre-Appell polynomials and related members.

3. Determinant forms and recurrence relations

To express the generalized Legendre-Appell polynomials via determinant, we prove the following result:

Theorem 3.1. *For the generalized Legendre-Appell polynomials ${}_{\nu S}A_n(x,y;\alpha)$, the following determinant form holds true:*

$${}_{\nu S}A_0(x,y;\alpha) = \frac{1}{\beta}_0, \tag{46}$$

$${}_{\nu S}A_n(x,y;\alpha) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} 1 & {}_{\nu}S_1(x,y;\alpha) & {}_{\nu}S_2(x,y;\alpha) & \cdots & {}_{\nu}S_{n-1}(x,y;\alpha) & {}_{\nu}S_n(x,y;\alpha) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{47}$$

where $n = 1,2,\cdots$; $\beta_0,\ \beta_1,\cdots,\beta_n \in \mathbb{R}$; $\beta_0 \neq 0$ and

$$\beta_n = -\frac{1}{A_0}\Big(\sum_{k=1}^{n}\binom{n}{k}A_k\ \beta_{n-k}\Big),\ n = 1,2,\cdots\ . \tag{48}$$

Proof. We consider the following determinant definition for the Appell polynomials [5, p.1533]:

$$A_0(y) = \frac{1}{\beta_0}, \tag{49}$$

$$A_n(y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & y & y^2 & \cdots & y^{n-1} & y^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}. \tag{50}$$

Taking $n = 0$ in formula (33) and then using equations (49) and (50), we get assertion (46).

Further, expanding determinant (50) with respect to the first row and then operating $\left(\alpha - \left(D_x^{-1}\frac{\partial^2}{\partial y^2}\right)\right)^{-\nu}$ on both sides of the resulting equation and then using equations (19) and (26), we find

$$
\begin{aligned}
{}_{\nu S}A_n(x,y;\alpha) = \frac{(-1)^n {}_{\nu}S_0(x,y;\alpha)}{(\beta_0)^{n+1}} &\begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} - \frac{(-1)^n {}_{\nu}S_1(x,y;\alpha)}{(\beta_0)^{n+1}} \\
&\begin{vmatrix} \beta_0 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \frac{(-1)^n {}_{\nu}S_2(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} + \cdots \\
&+ \frac{(-1)^{2n-1} {}_{\nu}S_{n-1}(x,y;\alpha)}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 \end{vmatrix}
\end{aligned}
$$

$$+\frac{{}_{\nu}S_n(x,y;\alpha)}{(\beta_0)^{n+1}}\begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \binom{n}{n-1}\beta_1 \end{vmatrix}. \tag{51}$$

Combining the terms in the r.h.s. of equation (51), we are lead to assertion (47).

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1, 2, ..., n)$ (for which the determinant form of the Appell polynomials $A_n(y)$ reduce to the Bernoulli polynomials $B_n(y)$ [4, 5]) in equations (46) and (47), the determinant definition of the generalized Legendre-Bernoulli polynomials ${}_{\nu S}B_n(x, y; \alpha)$ can be obtained.

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1, 2, ..., n)$ (for which the determinant form of the Appell polynomials $A_n(y)$ reduce to the Euler polynomials $E_n(y)$ [5] in equations (46) and (47), the determinant definition of the generalized Legendre-Euler polynomials ${}_{\nu S}E_n(x, y; \alpha)$ can be obtained.

• Taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2(i+1)}$ $(i = 1, 2, ..., n)$ (for which the determinant form of the Appell polynomials $A_n(y)$ reduce to the Genocchi polynomials $G_n(y)$ in equations (46) and (47), the determinant definition of the generalized Legendre-Genocchi polynomials ${}_{\nu S}G_n(x, y; \alpha)$ can be obtained.

Next, we derive the recurrence relations for the generalized Legendre-Appell polynomials ${}_{S}A_{n,\nu}(x, y; \alpha)$ by taking into consideration their generating equation.

On differentiating generating function (30), with respect to y, D_x^{-1} and α, we find the following recurrence relations for the generalized Legendre-Appell polynomials ${}_{\nu S}A_n(x, y; \alpha)$:

$$\begin{aligned} &\frac{\partial}{\partial y}\Big({}_{\nu S}A_n(x,y;\alpha)\Big) = n\ {}_{\nu S}A_n(x,y;\alpha), \\ &\frac{\partial}{\partial D_x^{-1}}\Big({}_{\nu S}A_n(x,y;\alpha)\Big) = \nu n(n-1)\ {}_{\nu+1 S}A_n(x,y;\alpha), \\ &\frac{\partial}{\partial \alpha}\Big({}_{\nu S}A_n(x,y;\alpha)\Big) = -\nu\ {}_{\nu+1 S}A_n(x,y;\alpha), \end{aligned} \tag{52}$$

which consequently yields

$$\frac{\partial}{\partial D_x^{-1}}\Big({}_{\nu S}A_n(x,y;\alpha)\Big) = -\frac{\partial^3}{\partial y^2 \partial \alpha}\ {}_{\nu S}A_n(x,y;\alpha). \tag{53}$$

In the next section certain identities for the Legendre-Appell polynomial family and some of its members are obtained as applications of the results derived in Section 2.

4. Applications

Several identities involving Appell polynomials and related members are known in literature. The operational formalism developed in the previous section can be used to obtain the corresponding identities involving generalized Legendre-Appell and related polynomials. To achieve this, we perform the following operation:

$(\mathcal{O})$ operating $\left(\alpha - D_x^{-1}\frac{\partial^2}{\partial y^2}\right)^{-\nu}$ on both sides of a given relation.

First, we consider the following results for the Appell polynomials $A_n(y)$ [5, (31-32) p.1534]:

$$A_n(y) = \frac{1}{\beta_0}\Big(y^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}\ A_k(y)\Big),\ n = 1, 2, \cdots\ , \tag{54}$$

$$y^n = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}\ A_k(y),\ n = 0, 1, \cdots\ . \tag{55}$$

Performing operation $(\mathcal{O})$ on both sides of the above equations and then using operational definitions (19) and (26), we obtain the following identities involving generalized Legendre-Appell polynomials ${}_{\nu S}A_n(x, y; \alpha)$:

$${}_{\nu S}A_n(x, y; \alpha) = \frac{1}{\beta_0}\Big({}_{\nu}S_n(x, y; \alpha) - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}\ {}_{\nu S}A_k(x, y; \alpha)\Big),\ n = 1, 2, \cdots\ , \tag{56}$$

$${}_{\nu}S_n(x, y; \alpha) = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}\ {}_{\nu S}A_k(x, y; \alpha),\ n = 0, 1, \cdots\ . \tag{57}$$

Next, we recall the following functional equations involving Bernoulli polynomials $B_n(y)$ [12, p.26]:

$$B_n(y + 1) - B_n(y) = n\ y^{n-1},\ n = 0, 1, 2.....\ , \tag{58}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}B_m(y) = ny^{n-1},\ n = 2, 3, 4.....\ , \tag{59}$$

$$B_n(my) = m^{n-1}\sum_{k=0}^{m-1}B_n\left(y + \frac{k}{m}\right),\ n = 0, 1.2, ...;\ m = 1, 2, 3....\ . \tag{60}$$

Again, performing operation $(\mathcal{O})$ on both sides of the above equations and then using operational definitions (19) and (37), the following identities involving generalized Legendre-Bernoulli polynomials ${}_{S}B_{n,\nu}(x, y; \alpha)$ are obtained:

$${}_{\nu S}B_n(x, y + 1; \alpha) - {}_{\nu S}B_n(x, y; \alpha) = n\ {}_{\nu}S_{n-1}(x, y; \alpha),\ n = 0, 1, 2\cdots, \tag{61}$$

$$\sum_{m=0}^{n-1}\binom{n}{m}{}_{\nu S}B_m(x, y; \alpha) = n\ {}_{\nu}S_{n-1}(x, y; \alpha),\ n = 2, 3, 4\cdots, \tag{62}$$

$${}_{\nu S}B_n(mx, my; \alpha) = m^{n-1}\sum_{k=0}^{m-1}{}_{\nu S}B_n\left(x, y + \frac{k}{m}; \alpha\right),\ n = 0, 1, 2, ...;\ m = 1, 2, 3\cdots\ . \tag{63}$$

Further, performing operation $(\mathcal{O})$ with use of operational rules (19), (40)and (43) on the following functional equations involving Euler polynomials $E_n(y)$ [12, p. 30] and Genocchi polynomials $G_n(y)$ [6, p. 1038, (42)]:

$$E_n(y + 1) + E_n(y) = 2y^n,$$

$$E_n(my) = m^n\sum_{k=0}^{m-1}(-1)^k E_n\left(y + \frac{k}{m}\right)\ n = 0, 1, 2...;\ m \text{ odd},$$

$$G_{n+1}(y) + G_n(y) = 2ny^{n-1},$$

yields the following identities involving the generalized Legendre-Euler polynomials ${}_{S}E_{n,\nu}(x, y; \alpha)$ and generalized Legendre-Genocchi polynomials ${}_{S}G_{n,\nu}(x, y; \alpha)$:

$${}_{\nu S}E_n(x, y + 1; \alpha) + {}_{\nu S}E_n(x, y; \alpha) = 2\ {}_{\nu}S_n(x, y; \alpha), \tag{64}$$

$${}_{\nu S}E_n(mx, my; \alpha) = m^n\sum_{k=0}^{m-1}(-1)^k{}_{\nu S}E_n\left(x, y + \frac{k}{m}; \alpha\right),\ n = 0, 1.2, ...;\ m \text{ odd}, \tag{65}$$

$${}_{\nu S}G_{n+1}(x, y; \alpha) + {}_{\nu S}G_n(x, y; \alpha) = 2n\ {}_{\nu}S_{n-1}(x, y; \alpha). \tag{66}$$

Finally, considering the following connection formulae involving the Bernoulli and Euler polynomials [12, pp. 29-30]:

$$B_n(y) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} B_{n-m} E_m(2y), \ n = 0, 1, 2... \ , \tag{67}$$

$$E_n(y) = \frac{2^{n+1}}{n+1} \Big[B_{n+1}\Big(\frac{y+1}{2}\Big) - B_{n+1}\Big(\frac{y}{2}\Big) \Big], \ n = 0, 1, 2... \ , \tag{68}$$

$$E_n(my) = -\frac{2^{m^n}}{n+1} \sum_{k=0}^{m-1} (-1)^k B_{n+1}\Big(\frac{y+k}{m}\Big), \ n = 0, 1, 2..., m \text{ even}, \tag{69}$$

which on performing operation ($\mathcal{O}$) and then using appropriate operational definitions yields the following connection formulae involving the generalized Legendre-Bernoulli and Legendre-Euler polynomials:

$${}_{\nu S}B_n(x, y; \alpha) = 2^{-n} \sum_{m=0}^{n} \binom{n}{m} B_{n-m} \ {}_{\nu S}E_n(2x, 2y; \alpha), \ n = 0, 1, 2... \ , \tag{70}$$

$${}_{\nu S}E_n(x, y; \alpha) = \frac{2^{n+1}}{n+1} \Big[{}_{\nu S}E_{n+1}\Big(\frac{x}{2}, \frac{y+1}{2}; \alpha\Big) - {}_{\nu S}E_{n+1}\Big(\frac{x}{2}, \frac{y}{2}; \alpha\Big) \Big], \ n = 0, 1, 2... \ , \tag{71}$$

$${}_{\nu S}E_n(mx, my; \alpha) = -\frac{2m^n}{n+1} \sum_{k=0}^{m-1} (-1)^k \ {}_{\nu S}B_{n+1}\left(\frac{x}{m}, \frac{y+k}{m}; \alpha\right), \ n = 0, 1, 2, ...; \ m \text{ even}. \tag{72}$$

Recently, certain new families of three variables associated with Legendre polynomials are introduced. This approach can be further extended to give several important results for these polynomials involving exponential and fractional operators.

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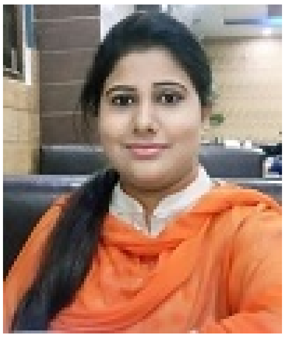

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